k-dimensional transversals for fat convex sets

Attila Jung

Eötvös Loránd University

June 25, 2025

Joint with Dömötör Pálvölgyi

Fractional Helly

Helly's theorem (1923):

$\mathcal{F}$: finite family of convex sets in $\mathbb{R}^d$
any $(d+1)$-tuple of sets can be hit with a point
$\implies$ the whole family can be hit with a point.

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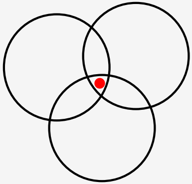
FH more precisely

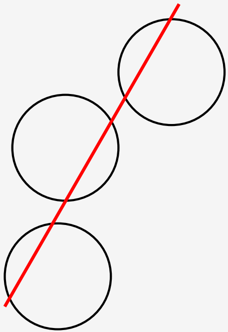
$\forall d \in \mathbb{N}, \alpha \in (0,1] \exists \beta \in (0,1]$ such that
if $\mathcal{F}$ is an n-element family of convex sets in $\mathbb{R}^d$
and $\alpha \binom{n}{d+1}$ of the $(d+1)$-tuples can be hit with a point
then there exists βn sets which can be hit with a point.

Vincensini's question

Vincensini '35:

Is it possible to generalize Helly's theorem to k-dimensional affine subspaces (k-flats) hitting convex sets?





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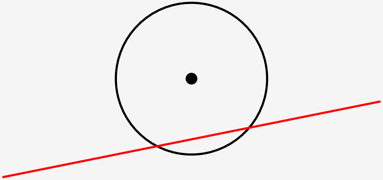

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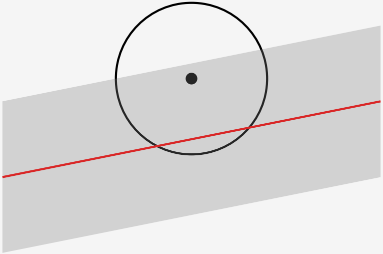

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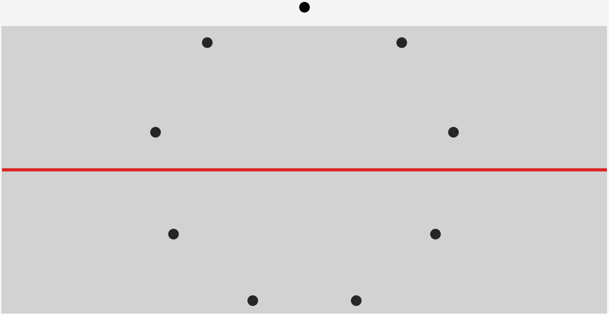

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J, Pálvölgyi '24+:

If $0 \leq k < d$, and $\mathcal{F}$ finite family of ρ-fat convex sets in $\mathbb{R}^d$, if a positive fraction of the $(k+2)$-tuples can be hit with a k-flat, then there exists a k-flat hitting a positive fraction of the family.

- ρ-fat: ratio of radii of min enclosing and max inscribed balls is at most ρ.

A spherical version

J, Pálvölgyi '24+:

If $0 \leq k < d$, and $\mathcal{F}$ finite family of ρ-fat convex sets in $\mathbb{S}^d$, if a positive fraction of the $(k+2)$-tuples can be hit with a great k-sphere, then there exists a great k-sphere hitting a positive fraction of the family.

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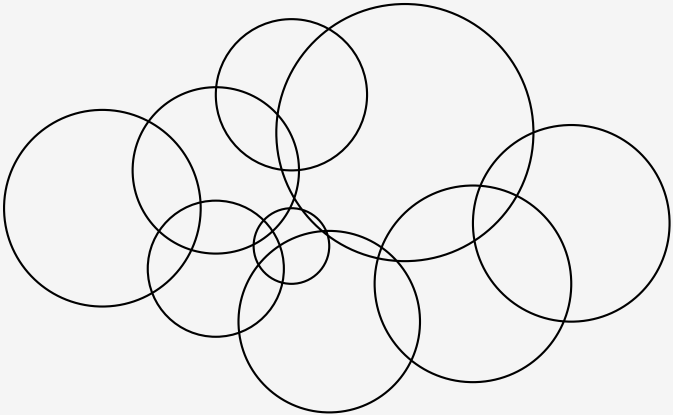
then there exists a great k-sphere hitting a positive fraction of the family.

- If $\mathbb{S}^d \subset \mathbb{R}^{d+1}$, and $L \subset \mathbb{R}^{d+1}$ is a linear $(k+1)$-flat, then $L \cap \mathbb{S}^d$ is a great k-sphere.

The $k = 0$ case



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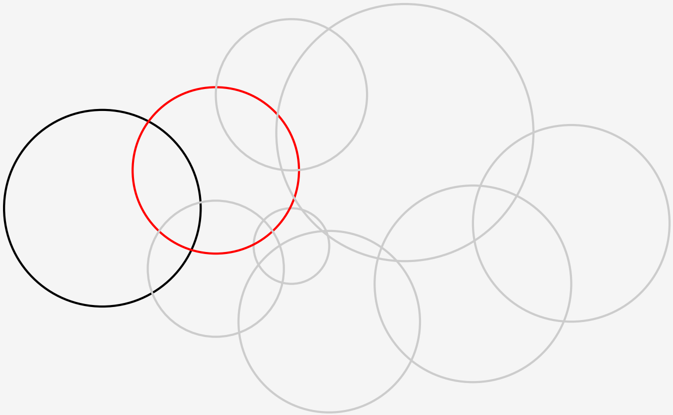

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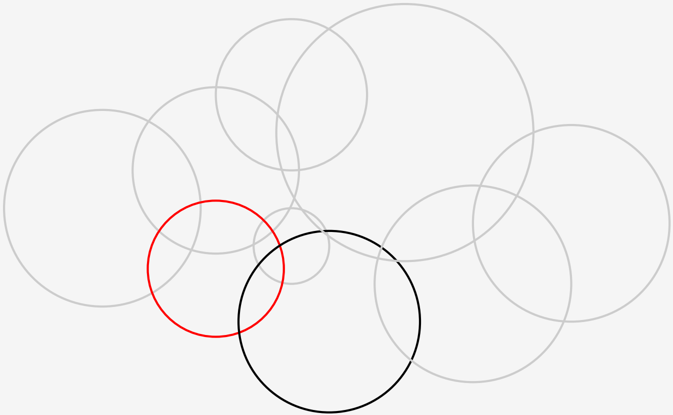

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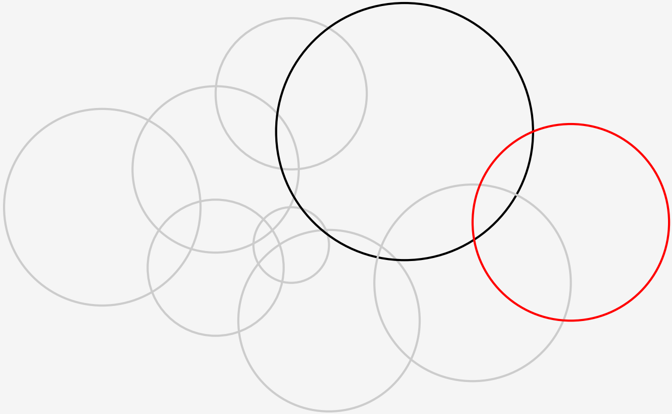

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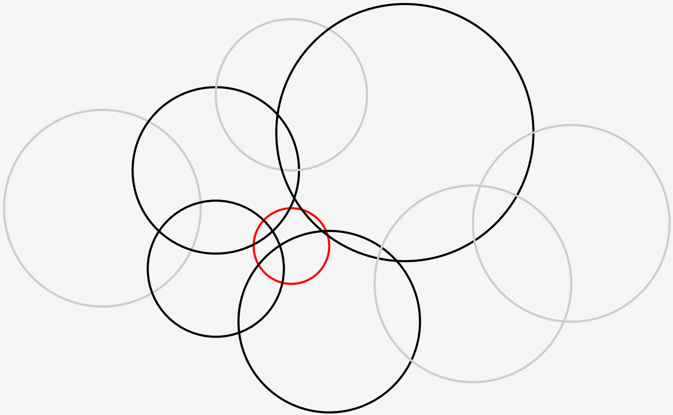

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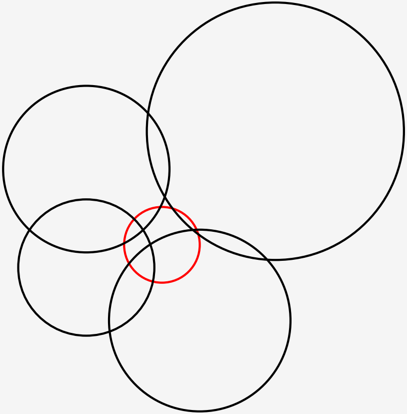

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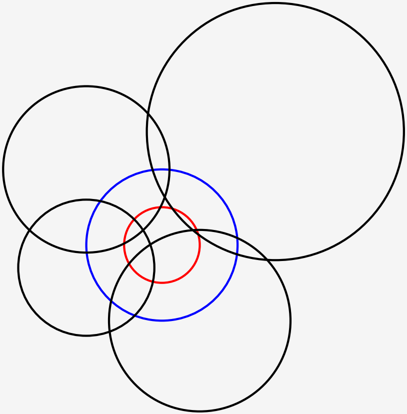
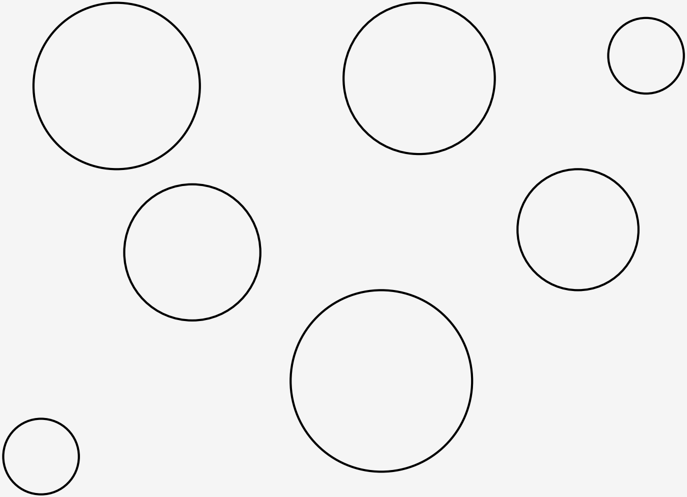


Induction step



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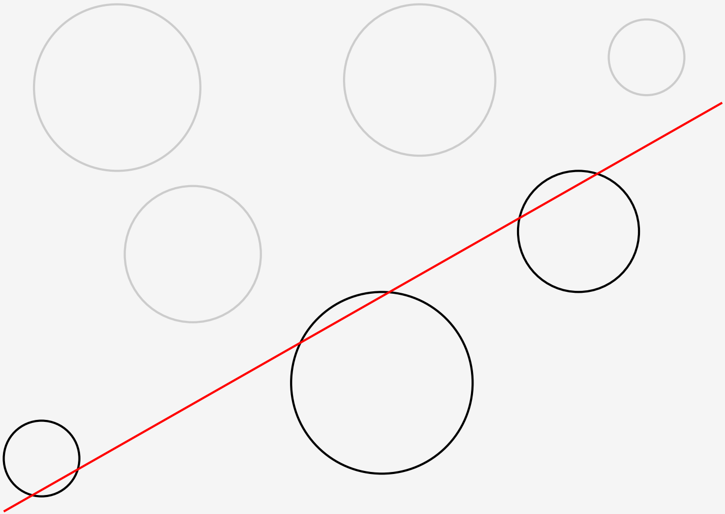

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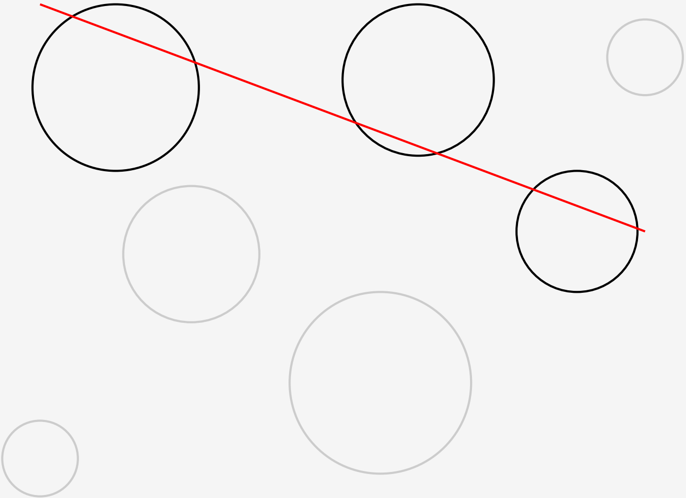

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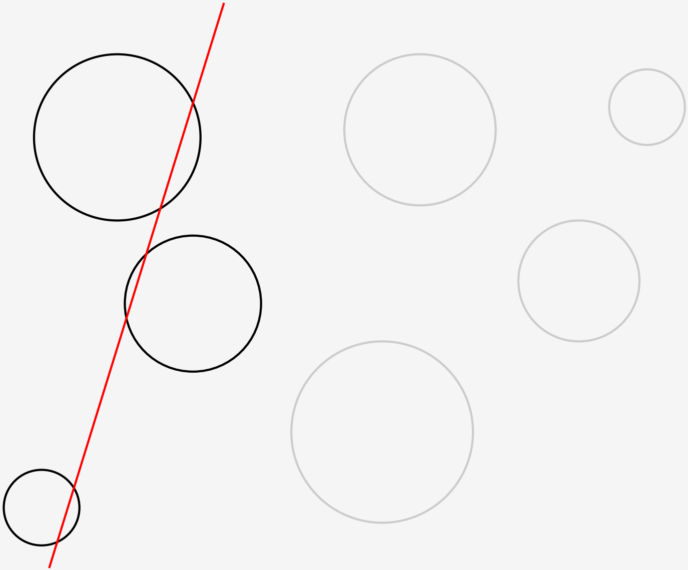

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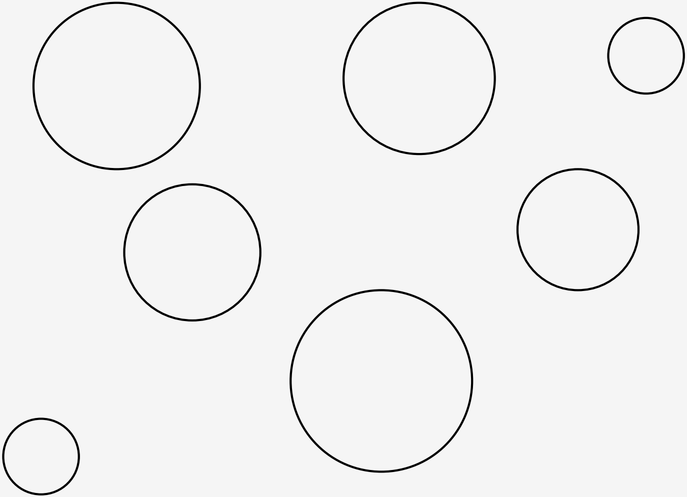

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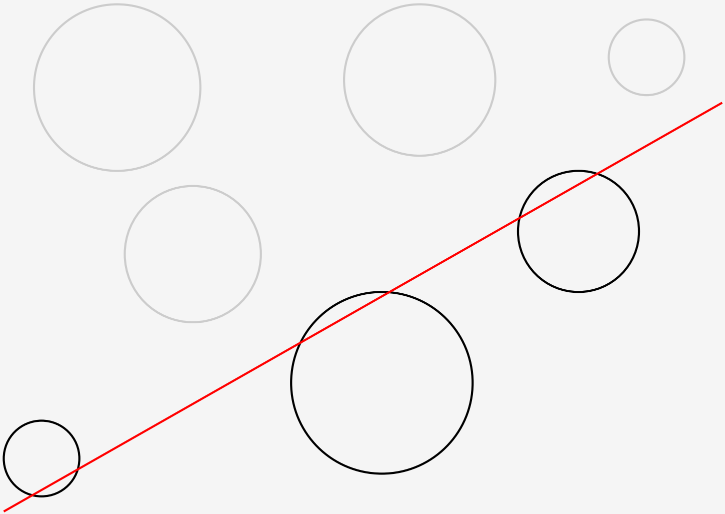

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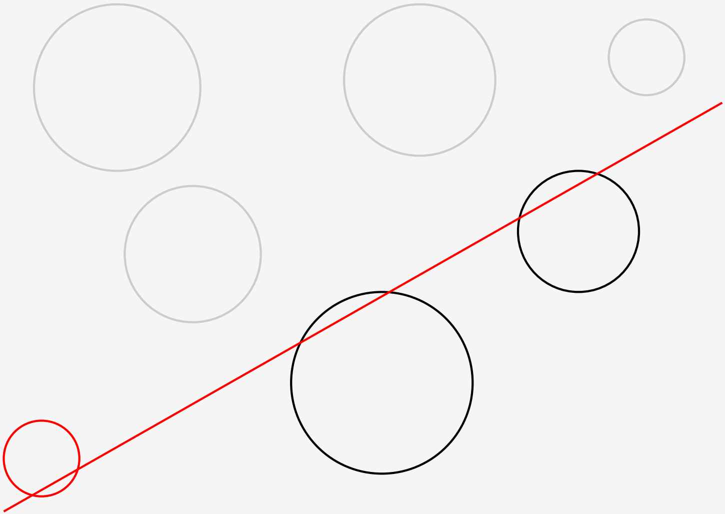

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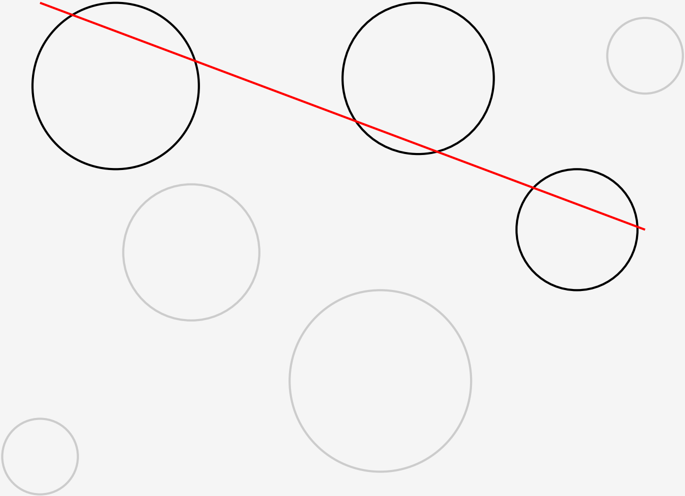

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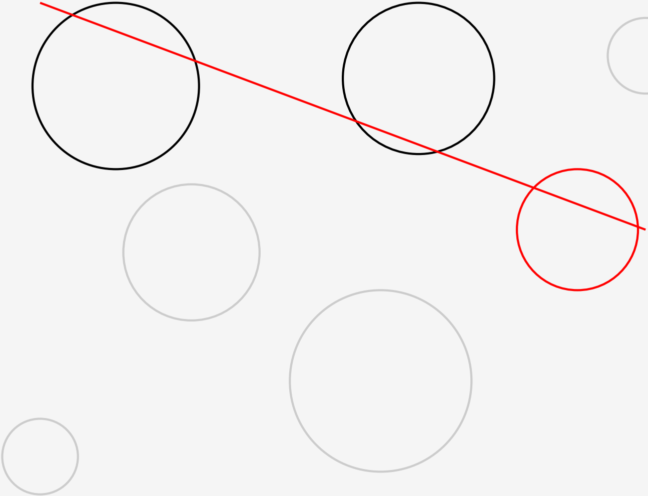

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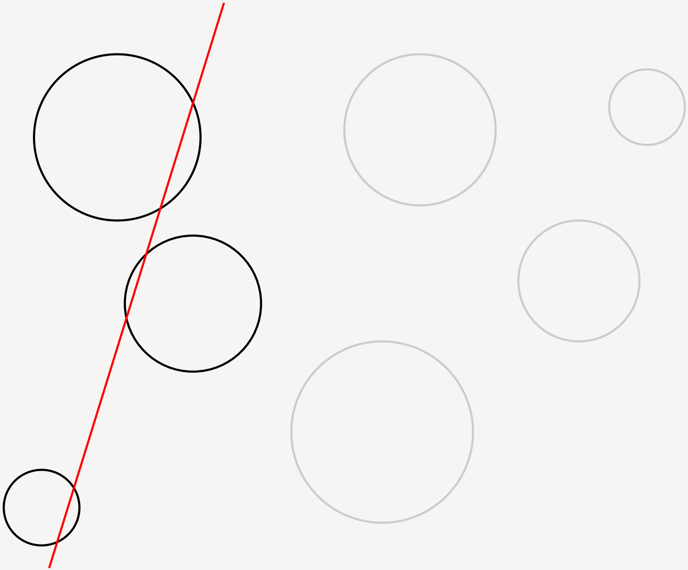

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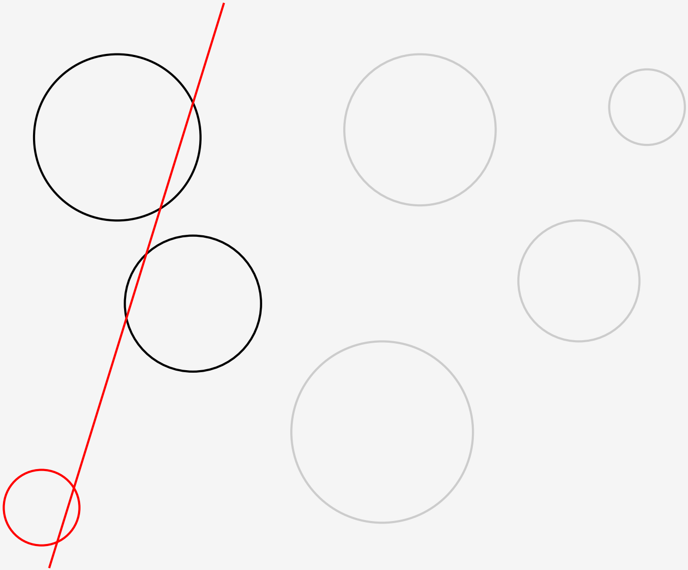

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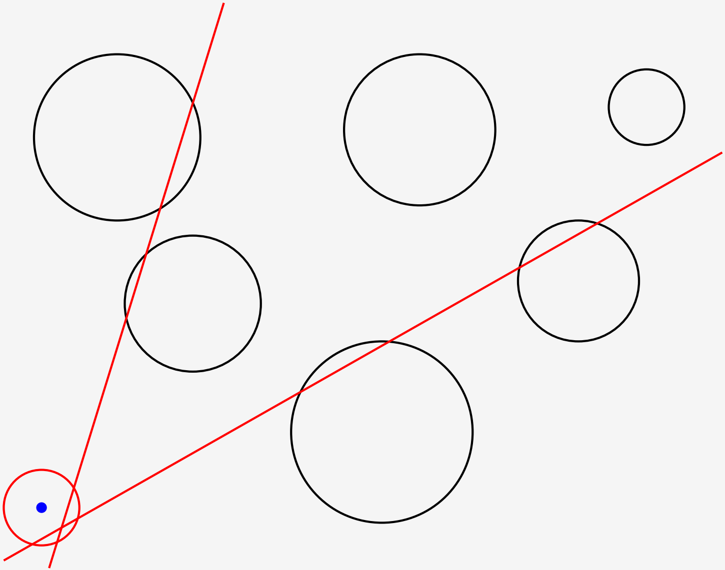

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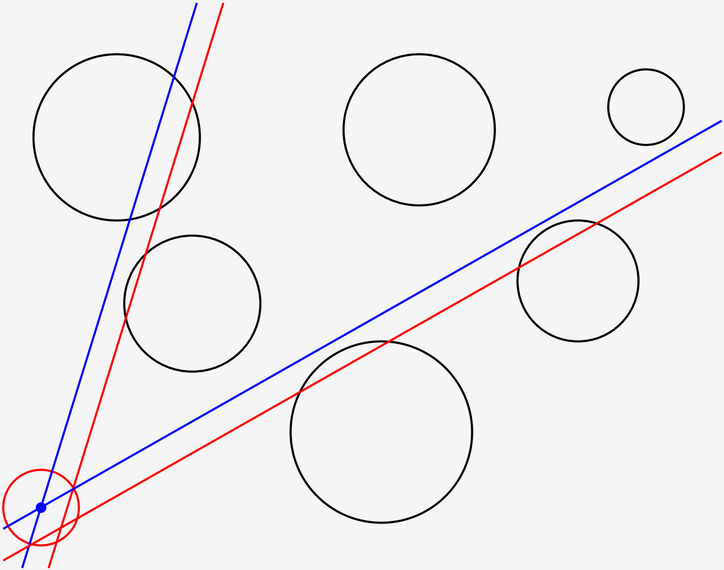

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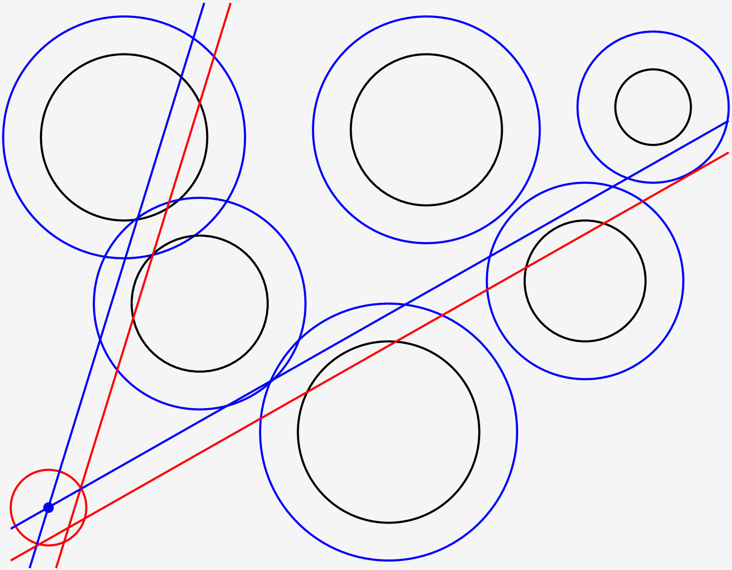

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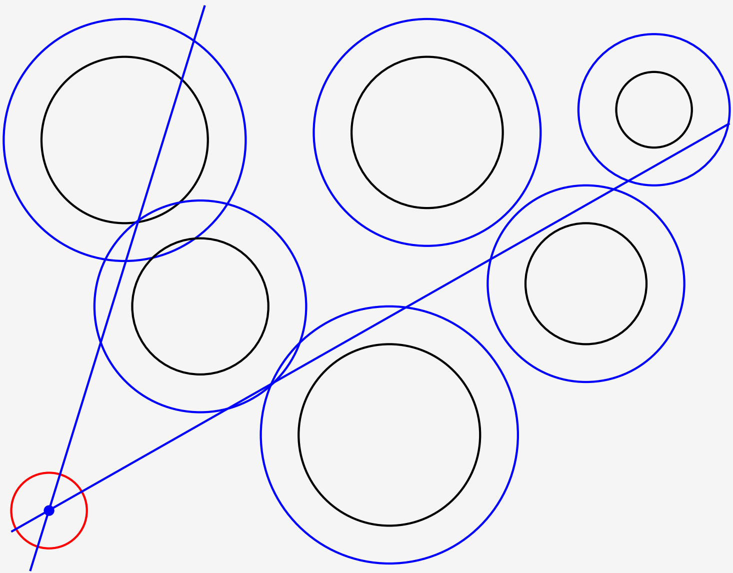

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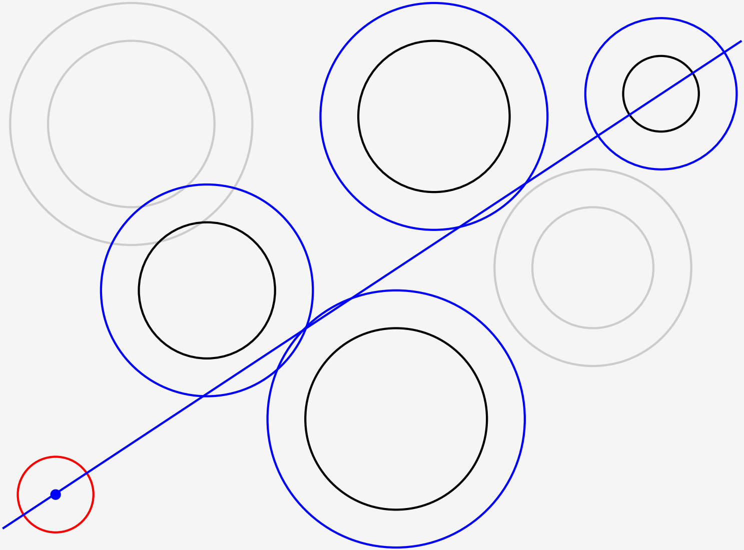

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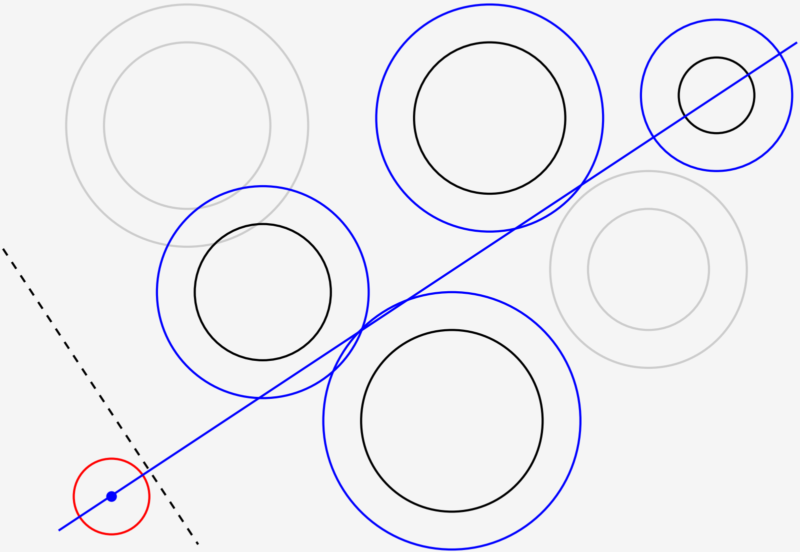

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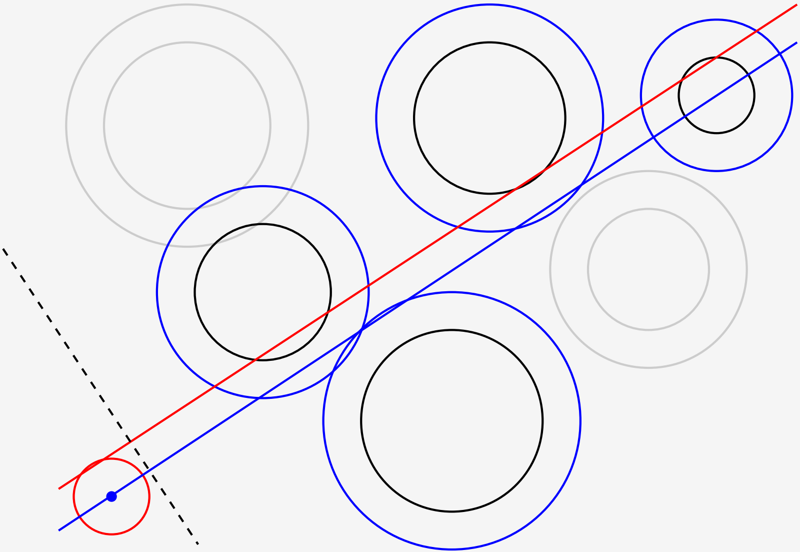

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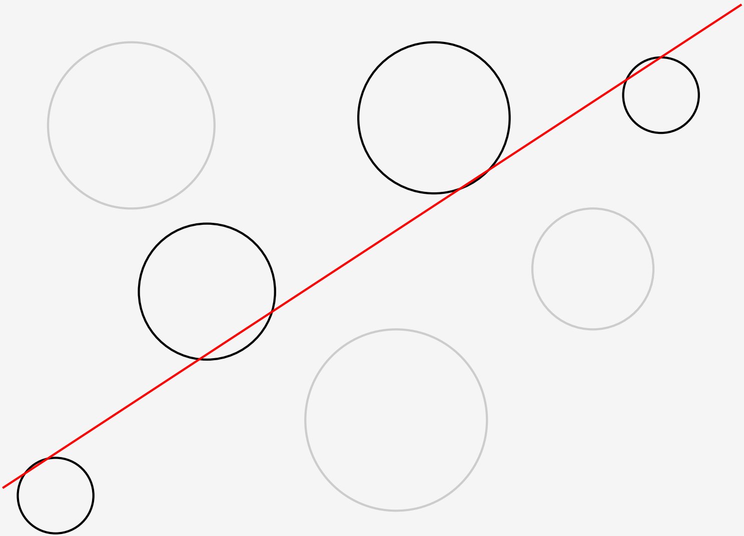

$(p, k+2)$-theorems

Conjecture

Let $p \geq q \geq k+2$, and $\mathcal{F}$ a finite family of ρ-fat convex sets in $\mathbb{R}^d$.
If among any p sets, some q can be hit with a k-flat,
then there is a family of $c(p, q, k) < \infty$ k-flats hitting all of $\mathcal{F}$.

We can only prove it for balls instead of ρ-fat convex sets.

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