Helly-type theorems for monotone properties of boxes

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Helly's theorem

Helly '23: Intersection of any $d+1$ members of a finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ is nonempty $\Rightarrow \cap\mathcal{F} \neq \emptyset$

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For boxes: Intersection of any 2 members of a finite family $\mathcal{B}$ of boxes in $\mathbb{R}^d$ is nonempty $\Rightarrow \cap\mathcal{B} \neq \emptyset$

Quantitative Helly

Bárány-Katchalski-Pach '82: Intersection of any $2d$ members of a family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$ has volume $\geq 1 \Rightarrow \mathrm{vol}(\cap \mathcal{F}) \geq \nu(d)$.

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For boxes: Intersection of any $2d$ members of a family $\mathcal{B}$ of boxes in $\mathbb{R}^d$ has volume $\geq 1 \Rightarrow \mathrm{vol}(\cap\mathcal{B}) \geq 1$.

A simple property of boxes

Lemma: Any finite family $\mathcal{B}$ of boxes in $\mathbb{R}^d$ contains a subfamily $\mathcal{B}' \subseteq \mathcal{B}$ of at most $2d$ boxes, such that $\cap\mathcal{B} = \cap\mathcal{B}'$.

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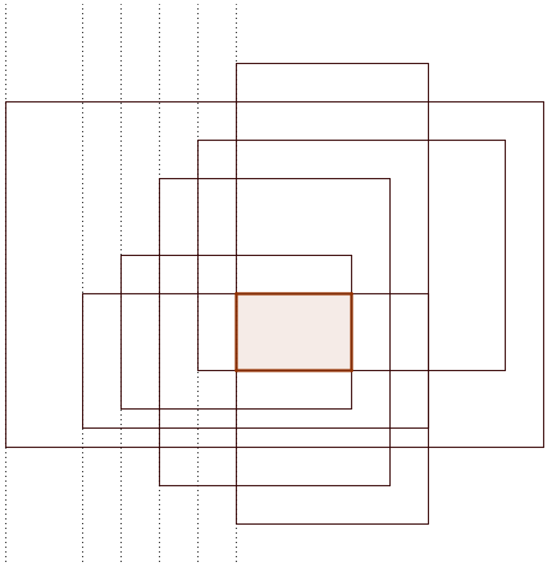
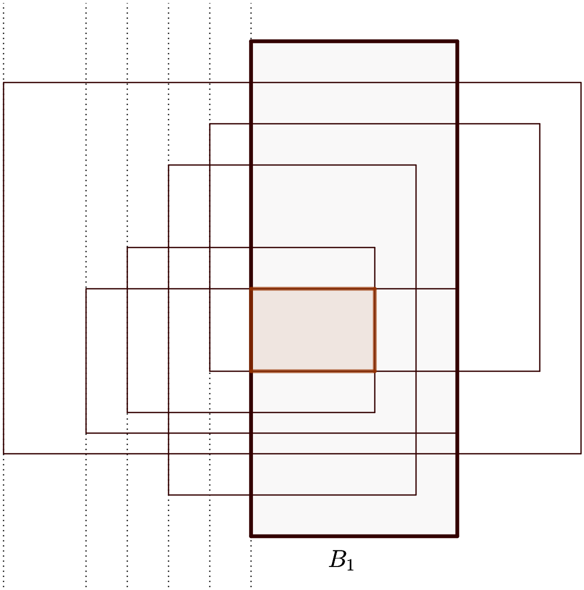
Proof.

$<_i$: ordering on $\mathcal{B}$ according to the bounding hyperplane in direction i.

$B_i :=$ smallest box in $\mathcal{B}$ according to $<_i$.

Then $\cap_i B_i = \cap\mathcal{B}$. □

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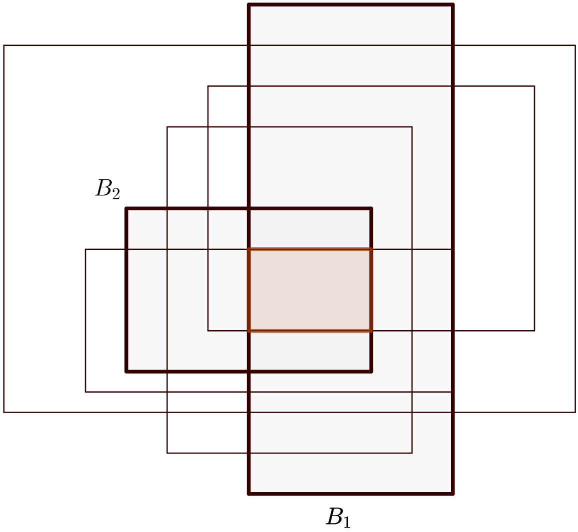
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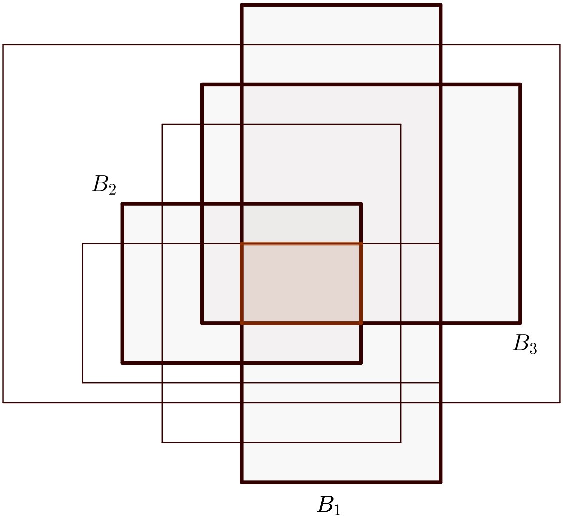

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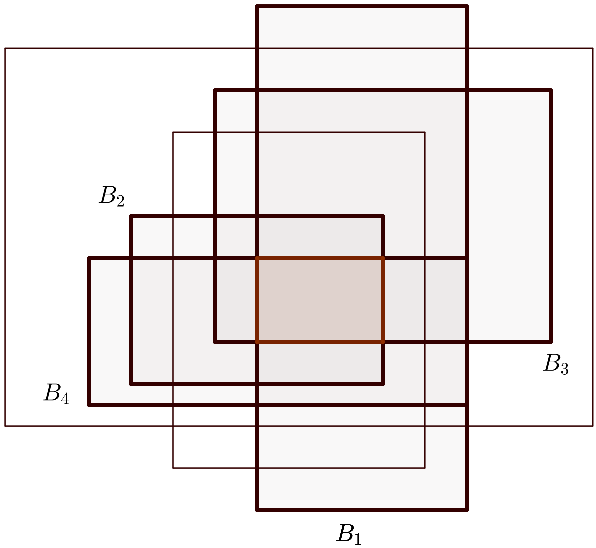

B_2
B_3
B_4
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- Halman '08 and Edwards-Soberón '24: when $P =$ containing a point from a given finite set

Colourful boxes

Theorem (FJ): Let $\mathcal{B}_1, \ldots, \mathcal{B}_{2d}$ be finite families of boxes in $\mathbb{R}^d$. Then there is a selection $B_i \in \mathcal{B}_i$ for each $i \in [2d]$ and an index $\ell \in [2d]$ such that $\cap_{i=1}^{2d} B_i \subset \cap \mathcal{B}_\ell$.

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Proof.

Sequentially define a permutation $\pi \in S_{2d}$ such that $\mathcal{B}_{\pi(i)}$ contains a minimal element $B_{\pi(i)}$ of $\cup_{j=1}^{n} \mathcal{B}_j \setminus (\cup_{j=1}^{i-1} \mathcal{B}_{\pi(j)})$ according to $<_i$.

Then $\cap_{i=1}^{2d} B_{\pi(i)} \subseteq \cap \mathcal{B}_{\pi(2d)}$, so we can choose $\{B_{\pi(1)}, \ldots, B_{\pi(2d)}\}$ and $\ell = \pi(2d)$. □

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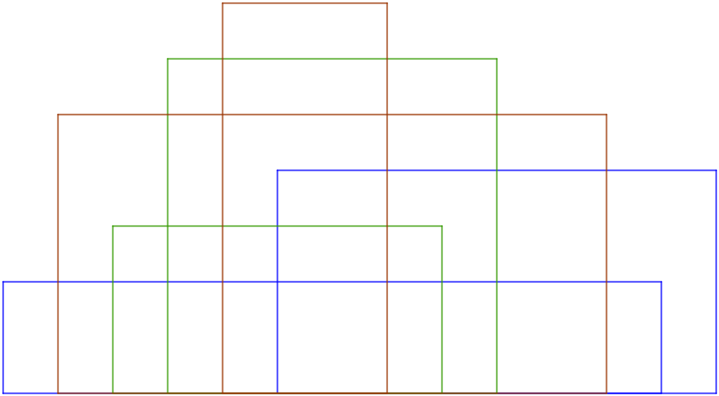

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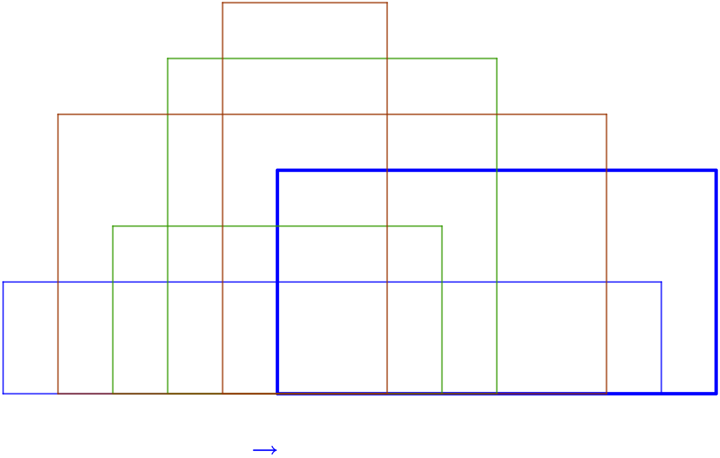

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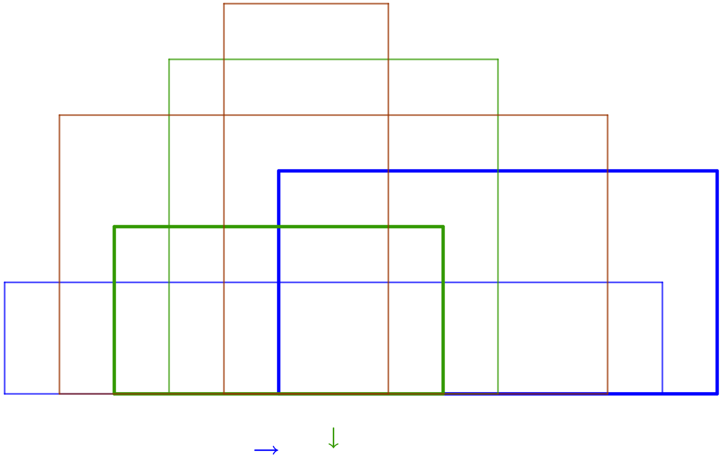

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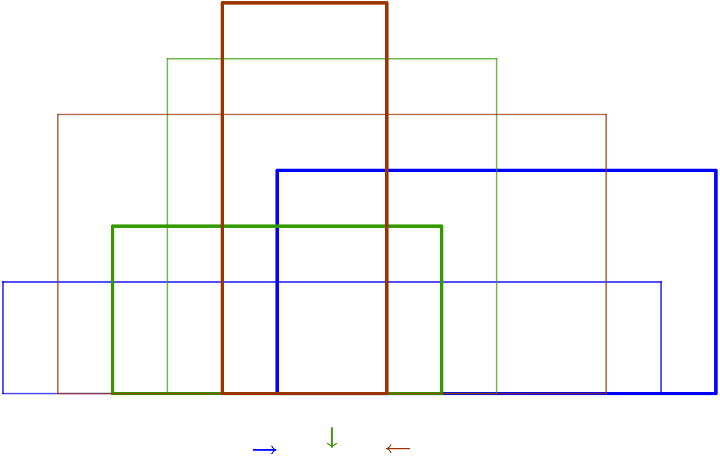

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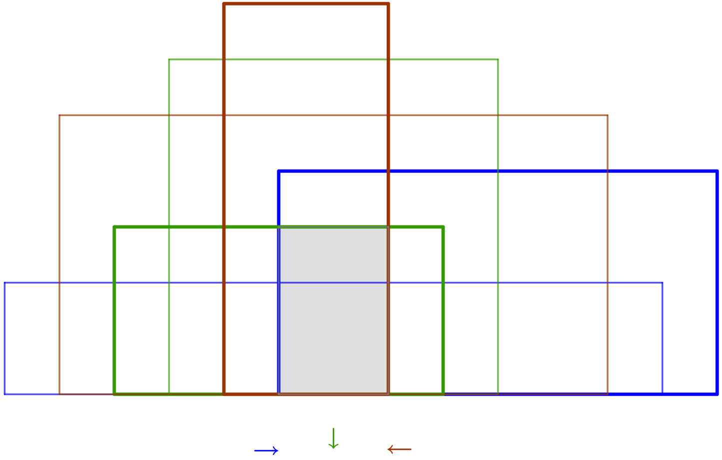

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Corollary: If for every choice $B_i \in \mathcal{B}_i$ the family $\{B_1, \ldots, B_{2d}\}$ is P-intersecting, then there exists an $\ell \in [2d]$ such that $\mathcal{B}_\ell$ is P-intersecting.

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- Edwards-Soberón '24: When $P =$ containing at least n points from a given finite set

Fractional versions

$\mathcal{B}$ is a family of boxes in $\mathbb{R}^d$.

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